Modified brane cosmologies with induced gravity, arbitrary matter content and a Gauss-Bonnet term in the bulk

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We extend the covariant analysis of the brane cosmological evolution in order to take into account, apart from a general matter content and an induced-gravity term on the brane, a Gauss-Bonnet term in the bulk. The gravitational effect of the bulk matter on the brane evolution can be described in terms of the total bulk mass as measured by a bulk observer at the location of the brane. This mass appears in the effective Friedmann equation through a term characterized as generalized dark radiation that induces mirage effects in the evolution. We discuss the normal and self-accelerating branches of the combined system. We also derive the Raychaudhuri equation that can be used in order to determine if the cosmological evolution is accelerating.

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I. INTRODUCTION

The cosmology of the Randall-Sundrum (RS) model with a single positive-tension brane [1] is a viable prototype for the evolution of a Universe identified with a hypersurface in a higher-dimensional non-compact space. The crucial property of the model that leads to the emergence of a realistic low-energy evolution is the effective compactification of gravity around the brane within the AdS background. In the simpler version [2] the matter is assumed to be localized on the brane, while the bulk space includes only a negative cosmological constant. The cosmological evolution can be identified with the motion of the brane within a static bulk [3].

An obvious generalization of this picture takes into account the possible presence of matter both in the bulk and on the brane, with the possibility of energy exchange [4, 5, 7]. It is remarkable that the general situation for an arbitrary bulk content has a very simple description [8, 9]. The effect of the bulk on the brane cosmological evolution, as determined by the effective Friedmann equation, can be incorporated in a single parameter characterizing the "strength" of the induced modifications: the integrated mass $\mathcal{M}$ of the bulk fluid as measured by a bulk observer at the location of the brane. If the spatial part of the brane has the geometry of a sphere, this mass is the same as the effective gravitational mass of the bulk. In the majority of cases the bulk observer is assumed to be comoving with the bulk fluid. For this reason we shall refer to the bulk mass as the *comoving mass*. We employ the same terminology for matter components, such as a bulk radiation fluid, for which there is no comoving observer. The comoving mass $\mathcal{M}$ in general depends on the brane scale factor R and the proper time τ on the brane. It appears in the effective Friedmann equation within a contribution $\sim \mathcal{M}(R,\tau)/R^4$ that has been termed generalized dark radiation [8, 9].

The next generalization includes terms that can be assumed to arise at the level of radiative corrections. The breaking of translational invariance by the location of the brane allows the presence of an induced gravity term on it. For a tensionless brane in a Minkowski bulk, one obtains the Dvali-Gabadadze-Porrati (DGP) model [10]. Specific examples of induced gravity can be obtained in string theory and are common in holographic descriptions [11, 12, 13] In the context of brane cosmology the presence of the induced-gravity term has a remarkable consequence: the appearance of a self-accelerating branch in the brane evolution [14, 15]. The existence of matter in the bulk can be taken into account in complete analogy to the RS case [16]. Its effects can be incorporated in the comoving mass $\mathcal{M}$ of the bulk. Exotic modifications of the brane cosmological evolution may arise [16]. Despite its very interesting properties, the self-accelerating branch is known to suffer from ghost-like instabilities [17, 18].

Radiative corrections in the bulk generate higher curvature terms. In particular, the Gauss-Bonnet (GB) combination is the leading bulk correction in the context of string theory [19]. The cosmological evolution in the presence of a bulk GB term (with or without induced gravity) has been discussed extensively [20, 21, 22, 23, 24]. An interesting feature is that it is possible to embed the brane so that it has self-accelerating cosmological expansion. Unfortunately, the branch of solutions that displays this behaviour is known to be unstable with respect to small perturbations [25] (see also [26, 27]). Finally, the combination of both the induced-gravity and GB terms leads to a multitude of cosmological scenarios, even in the absence of matter in the bulk [28].

We are interested in generalizing this framework in order to take into account an arbitrary matter content of the bulk. The effective action we consider has the form

$$S = \int d^5x\sqrt{-g}\left(\Lambda + M^3 R + \mathcal{L}^{mat}_{\text{BULK}} + M^3\alpha\mathcal{L}_{GB}\right) + + \int d^4x\sqrt{-g_4}\left(-V + \mathcal{L}^{mat}_{\text{BRANE}} + r_c M^3 R_4\right). \quad (1)$$

In the first integral, $-\Lambda$ is the bulk cosmological constant (in general we assume $\Lambda \geq 0$), R_{ABCD}, R_{AB} the Riemann and Ricci tensors, R the curvature scalar of the

5-dimensional bulk spacetime with metric g_{AB}, and

$$\mathcal{L}_{GB} = R^2 - 4R_{AB}R^{AB} + R_{ABCD}R^{ABCD} \tag{2}$$

the GB term with coupling constant α. In the second integral, V is the brane tension, $g_{\alpha\beta}$ the induced 4-dimensional metric on the brane, g_4 its determinant, R_4 the corresponding curvature scalar, and r_c the characteristic length scale of induced gravity. The pure gravity part of the action includes the standard Einstein term, along with terms that could arise through quantum corrections. The matter contributions are arbitrary, and the effective action incorporates possible quantum corrections in this sector. We assume, however, that the corresponding energy-momentum tensor is consistent with the underlying geometry. As a result, the framework we consider is very general: It corresponds to a generic low-energy effective action in the Einstein frame, involving possibly a multitude of fields, and includes the leading quantum corrections.

The Einstein Field Equations (EFE) take the form

$$G^A_{\ B} + \alpha H^A_{\ B} = \bar{G}^A_{\ B} = \frac{1}{2M^3}\left(T^A_{\ B} + \Lambda\delta^A_{\ B}\right), \tag{3}$$

with the energy-momentum (EM) tensor $T^A_{\ B}$ given by

$$T_{AB} = T^{\text{BULK}}_{AB} + \delta(\eta)\,\tau_{AB}. \tag{4}$$

The corrections to the EFE (3) originating in the GB term are represented by the Lovelock tensor

$$\begin{aligned} H_{AB} \;=\; & 2RR_{AB} - 4R_A^{\ K}R_{KB} - 4R^{KL}R_{AKBL} + \\ & +2R_A^{\ KLM}R_{BKLM} - \frac{1}{2}g_{AB}\mathcal{L}_{GB}. \end{aligned} \tag{5}$$

The term T^{BULK}_{AB} is the bulk matter contribution, while τ_{AB} is the contribution from the brane located at $\eta(x^A) = 0$ and has the form [29]

$$\tau_{\alpha\beta} = T^{\text{BRANE}}_{\alpha\beta} - Vg_{\alpha\beta} - 2r_cM^3\ G_{\alpha\beta} \tag{6}$$

where $T^{\text{BRANE}}_{\alpha\beta}$ is the brane Energy-Momentum (EM) tensor. We note that the presence of the induced 4-dimensional curvature term results in a contribution to the tensor $\tau_{\alpha\beta}$ proportional to the Einstein tensor $G_{\alpha\beta}$ on the brane.

The purpose of this paper is to provide the general form of the solution of the above equations in the case of a FRW brane, for which there exist 3-dimensional hypersurfaces $\mathcal{D}$ invariant under a six-dimensional group of isometries. It follows that the surfaces $\mathcal{D}$ have constant curvature, parametrized by the constant $k = 0, \pm 1$. As we have already mentioned, the solution depends on the comoving mass $\mathcal{M}$ of the bulk fluid, which is a function of the brane scale factor R and the proper time τ on the brane. The form of this function can be determined only within a specific model of the bulk dynamics. Such a model may involve several bulk fields that possibly interact with the brane, or may employ a description in terms of a bulk cosmological fluid with a certain equation of state. The form of $\mathcal{M}(R,\tau)$ is necessary for a detailed discussion of the cosmological evolution of the brane. On the other hand, the general properties of the evolution, such as the presence of acceleration, can be determined from the structure of the equations we shall derive. For this reason we shall not consider specific models in this paper. We postpone a detailed investigation of the role of the comoving mass in individual cases for future work. Furthermore and although one can relax the assumption of a Z_2-symmetry [30] for the sake of simplicity, we maintain the existence of the mirror symmetry around the location of the brane.

Throughout this paper the following conventions are used: the pair $(\mathcal{M}, \mathbf{g})$ denotes the 5D bulk spacetime manifold endowed with a Lorentzian metric of signature $(-,+,+,+,+)$, bulk 5D indices are denoted by capital latin letters $A, B, ... = 0, 1, 2, ..., 4$, greek letters denote brane indices $\alpha, \beta, ... = 0, 1, 2, 3$, and lower case latin letters indicate spatial 3D components.

II. 3-BRANE EMBEDDING IN A STATIC BULK

Before presenting general and covariant results for the brane evolution, it is instructive to analyze a class of cases in which the problem is tractable in specific coordinate systems. We assume that for a certain observer the bulk content can be described as a static fluid. This assumption allows the possibility of an arbitrary number of fields and relies only on the existence of an observer comoving with the bulk matter. Clearly, important physical situations, such as those that involve the propagation of electromagnetic or gravitational radiation, are excluded by our assumption. However, many interesting backgrounds, including generalized black-hole ones, are allowed.

In order for the embedding of a cosmological 3-brane to be possible, the spatial part of the metric must include a 3-space of constant curvature. The resulting metric can be cast in the form

$$ds^2 = -n^2(r)dt^2 + r^2 d\Omega_k^2 + b^2(r)dr^2. \tag{7}$$

The lhs of the EFE (3) take the form

$$\bar{G}^{\,0}_{\ 0} \;=\; \frac{3}{b^2}\frac{1}{r}\left(\frac{1}{r} - \frac{b'}{b}\right) - \frac{3k}{r^2} + \frac{12\alpha b'}{r^3 b^3}\left(\frac{1}{b^2} - k\right) \tag{8}$$

$$\begin{aligned} \bar{G}^{\,i}_{\ j} \;=\; & \frac{1}{b^2}\left[\frac{1}{r}\left(\frac{1}{r} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{1}{r}\right) + \frac{n''}{n}\right] - \\ & -\frac{k}{r^2} + \frac{4\alpha b'n'}{r^2 b^3 n}\left(\frac{3}{b^2} - k\right) \end{aligned} \tag{9}$$

$$\bar{G}^{\,4}_{\ 4} \;=\; \frac{3}{b^2}\frac{1}{r}\left(\frac{1}{r} + \frac{n'}{n}\right) - \frac{3k}{r^2} - \frac{12\alpha n'}{r^3 b^2 n}\left(\frac{1}{b^2} - k\right), \tag{10}$$

where the prime denotes a derivative with respect to r.

The general form of the bulk energy-momentum tensor for the above geometric setup is

$$T^{\text{BULK}}_{AB} = \text{diag}\left(-\rho, \mathrm{p}, \mathrm{p}, \mathrm{p}, p\right), \tag{11}$$

with the two pressures p, p not equal unless the bulk matter can be interpreted as a perfect fluid. The 00 component of (3) gives

$$\left(\frac{r^2}{b^2} - \frac{2\alpha}{b^4} + \frac{4k\alpha}{b^2}\right)' = 2kr + \frac{1}{3M^3} r^3(\Lambda - \rho), \tag{12}$$

whereas the combination of the 00 and 44 components results in

$$\frac{(bn)'}{bn}\left(1 - \frac{4\alpha}{r^2 b^2} + \frac{4\alpha k}{r}\right) = \frac{1}{6M^3}\, b^2 r\,(\rho + p). \tag{13}$$

The conservation of the bulk energy-momentum tensor can be written in the form

$$\frac{p'}{\rho + p} + \frac{3(p - \mathrm{p})}{r(\rho + p)} = -\frac{\left(6M^3\right)^{-1}(p + \Lambda) r^3 b^2 + krb^2 - r}{r^2 - 4\alpha b^{-2} + 4\alpha k}. \tag{14}$$

Because of the Bianchi identities, the set (12)-(14) completely describes the solution.

Integrating (12) we find

$$\frac{r^2}{b^2} - \frac{2\alpha}{b^4} + \frac{4k\alpha}{b^2} = kr^2 + \frac{\Lambda r^4}{12M^3} - \frac{\mathcal{M}(r)}{6\pi^2 M^3} + 2\alpha k^2, \tag{15}$$

where $\mathcal{M}(r)$ satisfies

$$\frac{d\mathcal{M}}{dr} = 2\pi^2 r^3 \rho \tag{16}$$

and corresponds to the *comoving mass* of the bulk fluid. The above equation has the solutions

$$\frac{1}{b^2} = \frac{r^2}{4\alpha} + k - \epsilon_1 \frac{r^2}{4\alpha}\sqrt{1 - \frac{2\alpha\Lambda}{3M^3} + \frac{4\alpha\mathcal{M}(r)}{3\pi^2 M^3 r^4}}, \tag{17}$$

with $\epsilon_1 = \pm 1$. For $\alpha \to 0$ the solution with $\epsilon_1 = 1$ reproduces the known expression in the absence of the GB term [5]. This expression includes a contribution $\sim \Lambda r^2$ arising from the bulk cosmological constant.

In the branch described by the solution with $\epsilon_1 = -1$, the leading contribution to $1/b^2$ for $\alpha \to 0$ is $\sim r^2/\alpha$. One expects behaviour similar to that arising from an effective bulk cosmological constant $\sim 1/\alpha$. A brane embedded in such a background can display self-accelerating cosmological expansion with constant $H^2 \sim 1/\alpha$. Despite its very interesting properties, this branch is known to be unstable with respect to small perturbations [25].

In order to analyze the cosmological evolution of the brane, we employ the Gaussian normal coordinate system in which the metric takes the form

$$ds^2 = -m^2(\tau,\eta)d\tau^2 + a^2(\tau,\eta)d\Omega_k^2 + d\eta^2, \tag{18}$$

with $m(\tau, \eta = 0) = 1$. Through an appropriate coordinate transformation

$$t = t(\tau,\eta), \qquad r = r(\tau,\eta) \tag{19}$$

the metric (7) can be written in the form of equation (18). We define $R(\tau) = a(\tau, \eta = 0)$. In the system of coordinates (t, r) of equation (7) the brane is moving, as it is located at $r = R(\tau)$. Hence [5]

$$\frac{\partial t}{\partial \tau} = \frac{1}{n(R)}\left[b^2(R)\dot{R}^2 + 1\right]^{1/2} \tag{20}$$

$$\frac{\partial t}{\partial \eta} = -\epsilon_2 \frac{b(R)}{n(R)}\dot{R} \tag{21}$$

$$\frac{\partial a}{\partial \tau} = \dot{R} \tag{22}$$

$$\frac{\partial a}{\partial \eta} = -\epsilon_2 \frac{1}{b(R)}\left[b^2(R)\dot{R}^2 + 1\right]^{1/2}, \tag{23}$$

where the dot denotes a derivative with respect to proper time and $\epsilon_2 = \pm 1$. The η-derivatives are evaluated for $\eta = 0^+$. The value of ϵ_2 determines the way the brane is embedded in the bulk space. As we have mentioned earlier, we impose a Z_2-symmetry around the brane. For a matter configuration that solves the EFE in an infinite bulk before the brane embedding, only the solution in the half-space and its mirror image are employed in the construction that includes the brane. The value of ϵ_2 determines which half-space is used [18]. A negative sign in the r.h.s. of equation (23) means that r decreases away from the brane. In the absence of induced gravity and a GB term, the brane has positive tension. The configuration is stable under small perturbations and the massless graviton is localized near the brane.

The bulk energy-momentum tensor at the location of the brane in the coordinate system (τ, η) is

$$T^{\text{BULK}}_{00} = \rho(R) + \left[\rho(R) + p(R)\right] b^2(R)\dot{R}^2 \tag{24}$$

$$T^{\text{BULK}}_{ii} = R^2 \mathrm{p}(R) \quad \text{(no summation)} \tag{25}$$

$$T^{\text{BULK}}_{44} = p(R) + \left[\rho(R) + p(R)\right] b^2(R)\dot{R}^2 \tag{26}$$

$$T^{\text{BULK}}_{04} = \epsilon_2 b(R)\dot{R}\left[b^2(R)\dot{R}^2 + 1\right]^{1/2}\left[\rho(R) + p(R)\right] \tag{27}$$

The sign of T^{BULK}_{04} indicates whether a brane observer detects inflow or outflow of energy. This sign is determined by the value of ϵ_2. For $\epsilon_2 = 1$ the volume of the bulk space, as well as the matter it contains, grow for increasing η. Conservation of energy requires that there is energy outflow from the brane. For $\epsilon_2 = -1$ the brane embedding is such that the bulk volume diminishes for increasing η. This is consistent with energy flowing into the brane from both sides.

The lhs of (3) near the brane ($\eta \to 0^{\pm}$) take the form

(no summation over repeated indices)

$$\bar{G}^{\,0}_{\,0} = -\frac{12\alpha a''a'^2}{a^3}+\frac{3a'^2}{a^2}+\frac{12\alpha a''\dot{a}^2}{a^3}-\frac{3\dot{a}^2}{a^2}+ +\frac{3a''}{a}+\frac{12\alpha k a''}{a^3}-\frac{3k}{a^2} \tag{28}$$

$$\bar{G}^{\,i}_{\,i} = -\frac{4\alpha m''(a')^2}{a^2}+\frac{(a')^2}{a^2}+\frac{2m'a'}{a}-\frac{8\alpha m'a''a'}{a^2}- -\frac{8\alpha(m')^2\dot{a}^2}{a^2}+\frac{4\alpha m''\dot{a}^2}{a^2}-\frac{\dot{a}^2}{a^2}-\frac{8\alpha\dot{a}'^2}{a^2}-\frac{k}{a^2}+ +\frac{2a''}{a}+\frac{4\alpha k m''}{a^2}+m''+\frac{16\alpha m'\dot{a}\dot{a}'}{a^2}+ +\frac{8\alpha a''\ddot{a}}{a^2}-\frac{2\ddot{a}}{a} \tag{29}$$

$$\bar{G}^{\,4}_{\,4} = -\frac{12am'(a')^3}{a^3}+\frac{12\alpha\ddot{a}(a')^2}{a^3}+\frac{3(a')^2}{a^2}+ +\frac{12\alpha m'\dot{a}^2a'}{a^3}+\frac{12\alpha k m'a'}{a^3}+\frac{3m'a'}{a}-\frac{3\dot{a}^2}{a^2}- -\frac{3k}{a^2}-\frac{12\alpha\dot{a}^2\ddot{a}}{a^3}-\frac{12\alpha k\ddot{a}}{a^3}-\frac{3\ddot{a}}{a} \tag{30}$$

$$\bar{G}^{\,4}_{\,0} = -\frac{12\alpha m'\dot{a}^3}{a^3}+\frac{12\alpha\dot{a}'\dot{a}^2}{a^3}+\frac{12\alpha(a')^2m'\dot{a}}{a^3}- -\frac{12\alpha k m'\dot{a}}{a^3}-\frac{3m'\dot{a}}{a}-\frac{12\alpha(a')^2\dot{a}'}{a^3}+ +\frac{12\alpha k\dot{a}'}{a^3}+\frac{3\dot{a}'}{a}, \tag{31}$$

with a prime now denoting a derivative with respect to η.

We consider a brane Universe containing a perfect fluid with an energy-momentum tensor

$$T^{\rm BRANE}_{AB}=\delta(\eta)a^2(\tau,\eta){\rm diag}\left[\frac{m^2(\tau,\eta)}{a^2(\tau,\eta)}\tilde{\rho},\tilde{p},\tilde{p},\tilde{p},0\right]. \tag{32}$$

Integrating the 00 and ii components of (3) on a small η interval around the brane and using (6) we obtain

$$\frac{a'_+}{a}\left\{1+4\alpha\left[H^2+\frac{k}{a^2}-\frac{1}{3}\left(\frac{a'_+}{a}\right)^2\right]\right\} = -\frac{1}{12M^3}\left[V+\tilde{\rho}-6r_cM^3\left(H^2+\frac{k}{a^2}\right)\right] \tag{33}$$

$$m'_+\left(1+4\alpha H^2+\frac{4k\alpha}{a^2}-\frac{4\alpha {a'_+}^2}{a^2}\right)+a'_+\left(8\alpha\frac{\ddot{a}}{a^2}+\frac{2}{a}\right) = \frac{1}{4M^3}\left[\tilde{p}-V+2r_cM^3\left(H^2+\frac{k}{a^2}+\frac{2\ddot{a}}{a}\right)\right] \tag{34}$$

From (33) and (23) it is straightforward to derive the effective Friedmann equation

$$\left(H^2+\frac{1}{b^2a^2}\right)\left[1+4\alpha\left(\frac{k}{a^2}+\frac{2}{3}H^2-\frac{1}{3b^2a^2}\right)\right]^2 = \frac{1}{144M^6}\left[V+\tilde{\rho}-6r_cM^3\left(\frac{k}{a^2}+H^2\right)\right]^2. \tag{35}$$

In the low-energy limit ($\tilde{\rho}, H, a^{-1}\to 0$) our choice of sign for a'_+ in (23) must be consistent with the rhs of (33). For example, for $\alpha=r_c=0$ a negative sign ($\epsilon_2=1$) for a'_+ is consistent with the negative sign in the rhs of (33) only if $V>0$. It is worth mentioning that the above choice is referred as the "normal branch" and *contains* the Randall-Sundrum model as a particular case. In addition, the positivity of the brane tension guarantees stability under small perturbations and graviton localization near the brane. The self-accelerating branch of the DGP model [10, 14, 15] has $\alpha=V=0$, $\tilde{\rho}, a^{-1}\to 0$ and $H^2\sim 1/r_c$. It is then apparent from (33) that $a'_+>0$ ($\epsilon_2=-1$). This branch is known to have ghost-like instabilities [17]. The effective Friedmann equation (35) results from squaring a'_+. As a result, it includes both the normal and self-accelerating branches of the DGP model.

We define the effective cosmological constant

$$\lambda=\frac{V^2}{4M^6}-\frac{1-\sqrt{1-\tilde{\Lambda}}}{\alpha}\left(2+\sqrt{1-\tilde{\Lambda}}\right)^2 \tag{36}$$

where $\tilde{\Lambda}=2\alpha\Lambda/3M^3$. The low-energy cosmological evolution can be determined by expanding the Friedmann equation (35) in a^{-1} and $\tilde{\rho}$. We assume that the cosmological constant has been tuned to zero ($\lambda=0$). In the normal branch (no self-acceleration) with $\epsilon_1=\epsilon_2=1$ we expect the standard behaviour $H^2\sim\tilde{\rho}$ for $V\neq 0$. The Friedmann equation becomes

$$H^2+\frac{k}{a^2}=\left(72M^6+16\alpha\Lambda M^3+6r_cVM^3\right)^{-1}\left[\frac{4}{\pi^2}\left(2+\sqrt{1-\tilde{\Lambda}}\right)M^3\frac{\mathcal{M}(a)}{a^4}+V\tilde{\rho}\right]=\frac{1}{6M^2_{\rm Pl}}\left(\tilde{\rho}+\tilde{\rho}_d\right). \tag{37}$$

The effective Planck constant is

$$M^2_{\rm Pl}=\left(1+\frac{2\alpha\Lambda}{9M^3}+\frac{r_cV}{12M^3}\right)\frac{12M^6}{V} \tag{38}$$

and the *generalized dark radiation* [9]

$$\tilde{\rho}_d=\frac{2+\sqrt{1-\tilde{\Lambda}}}{3}\frac{12M^3}{\pi^2V}\frac{\mathcal{M}(a)}{a^4}. \tag{39}$$

The latter quantity desribes a mirage energy density that affects the evolution without arising from a source on the brane [31]. In the last two expressions we have absorbed the dependence on α and r_c in terms that approach 1 for $r_c, \alpha \to 0$. It is remarkable that, even in the presence of GB and induced-gravity terms, the effect of the bulk matter in the low-energy limit can be absorbed in a mirage density term $\sim \mathcal{M}(R)/R^4$. More exotic low-energy behaviour can be observed in the self-accelerating branch.

A simple example of the cosmological evolution we described can be obtained if the bulk energy-momentum tensor is assumed to have the form (11) with $p = \mathrm{p}$. In order to avoid the appearance of metric singularities in equation (15) we assume that $k = 1$. If the equation of state $p = p(\rho)$ is known, the bulk metric can be completely determined. This background is a generalization in four spatial dimensions and for a negative cosmological constant of the conventional solution describing the interior of stars. For this reason it has been termed AdS-star in reference [5].

The presence of a GB term leads to quantitative modifications of the $\alpha = 0$ configuration, but its qualitative form remains the same. Moreover, for an equation of state of the form $p = w\rho^\gamma$ the asymptotic form of $\mathcal{M}(r)$ for large r is the same for all values of α. This means that the late-time cosmological evolution is given by equations (37)-(39) with $\mathcal{M}(R)$ as in reference [5].

III. COVARIANT STRUCTURE OF THE 5D GB BULK WITH A FRW BRANE

As was demonstrated in [8] the bulk geometry with a FRW brane can be seen as a 5-dimensional generalization of the inhomogeneous orthogonal family of Locally Rotationally Symmetric (LRS class II) spacetimes [32]. The orbits $\mathcal{D}$ of the six-dimensional multiply transitive group of isometries are maximally symmetric 3-dimensional hypersurfaces with spatial (constant) curvature determined by the value of $k = 0, \pm 1$. The rotational symmetry of the bulk manifold $\mathcal{M}$ results in several key features of its geometric structure that allow a simplified and unified treatment. To begin with, let us first note that, from a dynamical point of view, there are two different ways to choose the unit timelike vector field normal to the group trajectories: either adapted to the average fluid velocity u^A of the bulk matter configuration or to the prolongated brane observers $\tilde{u}^A$. In coordinate language, this freedom is related to the choice of a particular coordinate system. As we are interested in a bulk that is not empty, it is convenient to use the velocity of the bulk observers u^A ($u^A u_A = -1$) in what follows.

Using the standard 1+4 splitting of the 5D spacetime manifold the deformation of the timelike congruence u^A can be expressed in terms of the corresponding kinematical quantities

$$u_{A;B} = \Sigma_{AB} + \frac{\Theta}{4} h_{AB} - \dot{u}_A u_B = \upsilon_{AB} - \dot{u}_A u_B, \quad (40)$$

where $\Sigma_{AB} = \left(h_A^K h_B^L - \frac{1}{4} h^{KL} h_{AB}\right) u_{(K;L)}$, $\Theta = u^A_{;A}$, and $\dot{u}^A = u^A_{;B} u^B$ are the rate of shear tensor, the rate of expansion scalar, the vorticity tensor and the acceleration of the observers u^A, respectively, $h_{AB} = g_{AB} + u_A u_B$ is the projection operator perpendicularly to u^A, and υ_{AB} is the extrinsic curvature of the 4D spaces $\mathcal{S}$ normal to u^A. We recall that the fluid velocity u^A is orthogonal to the group orbits, so that the vorticity tensor is $\Omega_{AB} = 0$. This fact allows us to interpret h_{AB} as the metric of $\mathcal{S}$ and employ an appropriate covariant derivative inherent to these spaces

$$D_L P^{AB\ldots}{}_{IJ\ldots} \equiv h^A_R h^B_S \ldots h^T_I h^X_J \ldots h^K_L \left(P^{RS\ldots}{}_{TX\ldots}\right)_{;K} \quad (41)$$

for any tensor $P^{AB\ldots}{}_{IJ\ldots}$. The structural characteristics of $\mathcal{S}$ are described in terms of the kinematical quantities of u^A by using the Gauss equation

$$^4R_{ABCD} = h_A{}^K h_B{}^L h_C{}^M h_D{}^N R_{KLMN} + 2\upsilon_{A[D}\upsilon_{C]B}, \quad (42)$$

with $^4R_{ABCD}$ the curvature tensor of $\mathcal{S}$.

The assumption of maximal symmetry of the group orbits $\mathcal{D}$ implies the existence of a preferred spacelike direction e^A ($e^A e_A = 1$, $u^A e_A = 0$), that represents the local axis of symmetry with respect to which all the geometrical, kinematical and dynamical quantities are invariant. As a result, all the spacelike vector or traceless tensor fields which are *covariantly constructed* via the timelike vector field u^A can be written in terms of e^A [8]. In order to study the structure of the spacelike congruence of curves generated by the unit spacelike vector field e^A we proceed in complete analogy with the 1+4 decomposition. The starting point is to introduce the projection tensor:

$$\Pi_{AB} \equiv g_{AB} + u_A u_B - e_A e_B = h_{AB} - e_A e_B \quad (43)$$

$$\Pi_A{}^A = 3,\ \Pi_C{}^A \Pi_B{}^C = \Pi_B{}^A,\ \Pi_B{}^A e^B = \Pi_B{}^A u^B = 0 \quad (44)$$

which is identified with the associated metric of the 3D manifold $\mathcal{D}$ (the *screen space*) normal to the pair $\{u^A, e^A\}$ at any spacetime event. The geometric structure of $\mathcal{D}$ is analyzed by decomposing into irreducible kinematical parts the first covariant derivatives of the spacelike vector field e^A according to [33]

$$\begin{aligned} e_{A;B} &= \mathcal{T}_{AB} + \frac{\vartheta}{3}\Pi_{AB} + \mathcal{R}_{AB} + e'_A e_B - \dot{e}_A u_B + \\ &\quad + \Pi_B{}^C \dot{e}_C u_A + \left[2\Omega_{CB} e^C - N_B\right] u_A. \end{aligned} \quad (45)$$

Here

$$\vartheta = e_{A;B}\Pi^{AB} = e^A{}_{;A} + \dot{e}^A u_A \quad (46)$$

$$\mathcal{T}_{AB} = \Pi_A{}^K \Pi_B{}^L \left[e_{(K;L)} - \frac{1}{3}\vartheta \Pi_{KL} \right], \quad \mathcal{T}_{KL}\Pi^{KL} = 0 \tag{47}$$

$$\mathcal{R}_{AB} = \Pi_A{}^K \Pi_B{}^L e_{[K;L]} \tag{48}$$

$$N^A = \Pi_K{}^A \mathcal{L}_{\mathbf{u}} e^K \tag{49}$$

are the rate of the surface expansion, the rate of shear tensor, the rotation tensor and the Greenberg vector field of the spacelike congruence e^A, respectively. We use the notation

$$K'_{A\ldots} \equiv K_{A\ldots;L} e^L \tag{50}$$

for the directional derivative along the vector field e^A of any scalar or tensorial quantity.

Each of the above kinematical quantities carries information on the (overall or in different directions) distortion of $\mathcal{D}$ as measured by the bulk observers u^A. They have a similar interpretation as the corresponding quantities of the timelike congruence u^A. The new ingredient is the Greenberg vector N_A which represents the "coupling" mechanism between directions normal and parallel to the screen space $\mathcal{D}$. For example, the equation $N^A = 0$ implies that the pair of vector fields $\{u^A, e^A\}$ generates a 2-dimensional integrable submanifold of $\mathcal{M}$ and the spacelike congruence e^A is "comoving" ("frozen-in") along the worldlines of the fundamental observers u^A. In addition, it ensures that $\mathcal{T}_{AB}$ and $\mathcal{R}_{AB}$ lie in the screen space and the unit vector fields $\{e^A, u^A\}$ are orthogonal at any instant.

An important consequence of the preferred spacelike direction, or equivalently the induced three dimensional isotropy, is the fact that any, covariantly defined via u^A, spacelike and traceless tensor field lying in the screen space must vanish. This means that $N^A = 0 = \Pi_B{}^C \dot{e}_C$ and $\mathcal{T}_{AB} = 0 = \mathcal{R}_{AB}$, and that the first derivatives of the spacelike congruence take the form

$$e_{A;B} = \frac{\vartheta}{3}\Pi_{AB} + e'_A e_B - \dot{e}_A u_B = K_{AB} + e'_A e_B, \tag{51}$$

where $K_{AB} = \Pi_A{}^I \Pi_B{}^J e_{(I;J)}$ is the extrinsic curvature of the spacelike hypersurfaces $\mathcal{S}$.

The vanishings of the Greenberg vector and the vorticity tensor imply that $\mathcal{D}$ is an assemply of 3D hypersurfaces that mesh together to generate the *integrable* manifold $\mathcal{D}$, which is a submanifold of the observers' instantaneous rest space. Consequently, the fully projected (perpendicular to the pair $\{u^A, e^A\}$) covariant derivative "$\|$" defined as

$$P^{AB\ldots}{}_{IJ\ldots\|L} \equiv \Pi_R{}^A \Pi_S{}^B \ldots \Pi_I{}^T \Pi_J{}^X \ldots \Pi_L{}^K \left(P^{RS\ldots}{}_{TX\ldots}\right)_{;K}, \tag{52}$$

represents the proper 3D covariant derivative, since $\Pi_{AB\|C} = 0$ and $A_{\|[KL]} = 0$ for any scalar quantity A.

The definition of the overall expansion ϑ of the spacelike congruence permits us to introduce the quantity ℓ according to

$$\vartheta = e^A{}_{\|A} \equiv 3\frac{\ell'}{\ell}. \tag{53}$$

Equation (53) makes clear the geometrical role of ℓ as the *average length scale* of $\mathcal{D}$. For example, in the spherically symmetric case $k = 1$ it represents the radius of the spheres $\mathcal{D}$. On the other hand the temporal $(u-)$change of ℓ is controlled by the expansion rate of the timelike congruence as measured in the screen space $\mathcal{D}$, namely

$$\Pi^{AB} u_{A;B} = u^A{}_{\|A} = 3\frac{\dot{\ell}}{\ell}. \tag{54}$$

Taking into account the above considerations, the length ℓ completely determines the volume of $\mathcal{D}$ which scales $\sim \ell^3$ as the screen space $\mathcal{D}$ evolves.

Regarding the dynamics, the matter content of the bulk is described by the energy-momentum T^{BULK}_{AB}, which can be written in the usual way with respect to the observers u^A

$$T^{\text{BULK}}_{AB} = \rho u_A u_B + p h_{AB} + 2q_{(A}u_{B)} + \pi_{AB}. \tag{55}$$

The dynamical quantities measured by the bulk observers are defined as

$$\begin{aligned} \rho &= T^{\text{BULK}}_{AB} u^A u^B, \; p = \frac{1}{4} T^{\text{BULK}}_{AB} h^{AB}, \; q_A = -h^C_A T^{\text{BULK}}_{CD} u^D, \\ \pi_{AB} &= h^C_A h^D_B T^{\text{BULK}}_{CD} - \frac{1}{4}(h^{CD} T^{\text{BULK}}_{CD}) h_{AB}. \end{aligned} \tag{56}$$

Because of the specific geometrical background of the bulk, it will be helpful to investigate the influence of the matter content on the curvature of the 3-dimensional screen space $\mathcal{D}$. This can be achieved by using the fact that the screen space $\mathcal{D}$ forms an integrable submanifold of $\mathcal{M}$ with a well defined metric Π_{AB} and a proper covariant derivative "$\|$". Then the corresponding Gauss equation for the distribution normal to the $1-$form $\mathbf{u}$ reads

$${}^3R_{ABCD} = \Pi_A{}^I \Pi_B{}^J \Pi_C{}^K \Pi_D{}^L \; {}^4R_{IJKL} + 2K_{A[C}K_{D]B}, \tag{57}$$

where ${}^3R_{ABCD}$ is the curvature tensor of the screen space $\mathcal{D}$. Contracting twice equation (57) and using (42) we obtain

$$\frac{{}^3R}{6} = \frac{1}{6}\Pi^{AC}\Pi^{BD} R_{ABCD} - \left(\frac{1}{3}\Pi^{AB} u_{A;B}\right)^2 + \left(\frac{1}{3}\vartheta\right)^2, \tag{58}$$

or equivalently in terms of the average scale factor and the 5D Einstein tensor,

$$\begin{aligned} \frac{k}{\ell^2} &= -\frac{1}{3}\left(\mathcal{E} + \frac{1}{2} G_{AB} g^{AB} - 2G_\perp\right) - \\ &\quad - \left(\frac{\dot{\ell}}{\ell}\right)^2 + \left(\frac{\ell'}{\ell}\right)^2. \end{aligned} \tag{59}$$

Here $\mathcal{E} = C_{ACBD}u^A e^C u^B e^D$ is the spatial eigenvalue of the electric part of the Weyl tensor and $3G_\perp \equiv \Pi^{AB}G_{AB}$. Equation (59) shows how the scalar curvature of the 3-space $\mathcal{D}$ is affected by the kinematics and the dynamical (when the 5D EFE are employed) content of the spacetime. Equivalently it represents the evolution equation of the average length scale.

We point out that equation (59) is a *first integral* of the propagation equation (along u^A) of $\dot{\ell}/\ell$, or the spatial variation (along e^A) of the spacelike expansion ϑ. In the context of brane cosmology, the physically interesting quantity is $\dot{\ell}/\ell$ which (*on* the brane) corresponds to the overall expansion H of the 3-brane. The expansion of the timelike congruence u^A is written

$$\frac{\Theta}{4} = \frac{\dot{\ell}}{\ell} + \frac{1}{3}\Sigma_{AB}e^A e^B. \tag{60}$$

Then, the temporal projection of the trace and traceless symmetric part of the Ricci identities for u^A gives evolution equations for Θ and Σ_{AB}. Combining the resulting expressions with the propagation of (60) we get

$$\begin{aligned}\left(\frac{\dot{\ell}}{\ell}\right)^{\cdot} + \left(\frac{\dot{\ell}}{\ell}\right)^2 &= -\frac{\ell'}{\ell}\dot{e}_A u^A - \frac{1}{3}G_{AB}e^A e^B + \\ &+ \frac{1}{3}\left(\mathcal{E} + \frac{1}{2}G_{AB}g^{AB} - 2G_\perp\right). \end{aligned}\tag{61}$$

Obviously, equation (61) is equivalent to the Raychaudhuri equation for the expansion H of the brane Universe.

If the GB correction is absent ($\alpha = 0$) the first term in the rhs of equation (59) can be determined by using the full 5D EFE (3) and the bulk energy momentum tensor (55). The result is [8]

$$\frac{k}{\ell^2} = \frac{\mathcal{M}}{6M^3\pi^2\ell^4} - \frac{\Lambda}{12M^3} - \left(\frac{\dot{\ell}}{\ell}\right)^2 + \left(\frac{\ell'}{\ell}\right)^2 \tag{62}$$

$$\begin{aligned}\left(\frac{\dot{\ell}}{\ell}\right)^{\cdot} + \left(\frac{\dot{\ell}}{\ell}\right)^2 &= -\frac{\ell'}{\ell}\dot{e}_A u^A - \frac{\mathcal{M}}{6M^3\pi^2\ell^4} - \frac{\Lambda}{12M^3} - \\ &- \frac{1}{6M^3}p_\parallel \end{aligned}\tag{63}$$

where $p_\parallel = T^{\text{BULK}}_{AB}e^A e^B$ is the pressure in the direction of the preferred spacelike vector field and $\mathcal{M}$ the *comoving mass* of the bulk fluid, satisfying

$$(\mathcal{M} - \mathcal{M}_0)' = 2\pi^2\rho\ell^3\ell'. \tag{64}$$

Only in the spherically symmetric case ($k = 1$) the comoving mass $\mathcal{M}$ has the usual physical interpretation of the effective gravitational mass contained within a spherical shell with radii ℓ_0 and ℓ. However, we shall refer to $\mathcal{M}$ as the integrated mass for all geometries of the hypersurfaces $\mathcal{D}$. The integration "constant" $\mathcal{M}_0$ in equation (64) can be interpreted as the mass of a black hole at $\ell_0 = 0$.

When $\alpha \neq 0$ the methodology breaks down because the EFE (3) include the Lovelock tensor as an additional contribution. Nevertheless, one can view

$$\hat{T}_{AB} = \frac{1}{2M^3}T^{\text{BULK}}_{AB} - \alpha H_{AB} \tag{65}$$

as an effective "energy-momentum" tensor, so that the EFE take their standard form and the methodology of [8] can be applied. This leads to

$$\mathcal{E} = -\frac{1}{2M^3}\left\{\frac{1}{2}\left(\hat{T} - 4\hat{p}_\perp\right) + \frac{\hat{\mathcal{M}}}{\pi^2\ell^4}\right\}, \tag{66}$$

where

$$\hat{T} = T^{\text{BULK}} - 2M^3\alpha\left(-\frac{1}{2}\mathcal{L}_{GB}\right) \tag{67}$$

$$\hat{p}_\perp = p_\perp - 2M^3\alpha\frac{1}{3}\Pi^{AB}H_{AB} \tag{68}$$

$$\left(\hat{\mathcal{M}} - \hat{\mathcal{M}}_0\right)' = 2\pi^2\left(T_{AB} - 2M^3\alpha H_{AB}\right)u^A u^B\ell^3\ell'. \tag{69}$$

It follows that the induced evolution of the average length scale ℓ is

$$\left(\frac{\dot{\ell}}{\ell}\right)^2 - \left(\frac{\ell'}{\ell}\right)^2 + \frac{k}{\ell^2} + \alpha\frac{\mathcal{M}_H}{3\ell^4} = -\frac{\Lambda}{12M^3} + \frac{\mathcal{M}}{6M^3\pi^2\ell^4}, \tag{70}$$

where

$$\mathcal{M}'_H = 2H_{AB}u^A u^B\ell^3\ell' = \frac{2}{3}H_{AB}u^A u^B\ell^4\vartheta. \tag{71}$$

Even though we have managed to express the eigenvalue of the electric part of the 5D Weyl tensor explicitly in terms of the "dynamical" variables of the effective bulk energy-momentum tensor (65), the appearance of the Lovelock correction term $\mathcal{M}_H$ makes the interpretation and the analysis of (70) unclear. It should be noted that, even though $\mathcal{M}_H$ is defined in a similar manner as the comoving mass $\mathcal{M}$ of the bulk fluid, the character of the former is purely *geometrical* since it contains only terms quadratic in the 5D curvature. In particular, because of equations (42) and (57) the 5D R_{ABCD}, R_{AB} and R can be expressed in terms of the 3D curvature quantities, the extrinsic curvatures υ_{AB}, K_{AB} and their first derivatives along e^A. Consequently, the Lovelock tensor is written as a quadratic expression involving ${}^3R$, υ_{AB}, K_{AB}. (We recall that the screen space $\mathcal{D}$ is maximally symmetric, so that the scalar ${}^3R$ completely determines the curvature tensor.)

After a tedious calculation and using equation (71) we get

$$\mathcal{M}_H = 6\ell^4\left[\left(\frac{\dot{\ell}}{\ell}\right)^2 - \left(\frac{\ell'}{\ell}\right)^2 + \frac{k}{\ell^2}\right]^2. \tag{72}$$

Defining the quantity

$$\mathcal{A}=\left(\frac{\dot{\ell}}{\ell}\right)^2-\left(\frac{\ell'}{\ell}\right)^2+\frac{k}{\ell^2}, \qquad (73)$$

we finally obtain

$$\mathcal{A}+2\alpha\mathcal{A}^2=-\frac{\Lambda}{12M^3}+\frac{\mathcal{M}}{6M^3\pi^2\ell^4}. \qquad (74)$$

Clearly, the last equation coincides (*off* the brane) with the equation of motion of the sheets of the bulk matter configuration in the 5D GB gravity. The solutions of the quadratic equation (74) are

$$\mathcal{A}=-\frac{1}{4\alpha}+\epsilon_1\frac{1}{4\alpha}\left(1-\frac{2\alpha\Lambda}{3M^3}+\frac{4\alpha\mathcal{M}}{3M^3\pi^2\ell^4}\right)^{1/2}, \qquad (75)$$

where $\epsilon_1=\pm 1$ is the same as the one defined in equation (17).

We conclude this section by noticing that with the above identifications, equation (63) is written as

$$\begin{aligned}\left(\frac{\dot{\ell}}{\ell}\right)^{\cdot}+\left(\frac{\dot{\ell}}{\ell}\right)^2 &= -\frac{\ell'}{\ell}\dot{e}_A u^A-\mathcal{A}-\frac{1}{6M^3}p_{\parallel}+\\ &-\frac{\Lambda}{6M^3}+\frac{1}{3}\alpha H_{AB}e^Ae^B. \qquad (76)\end{aligned}$$

Similarly as before, we can argue that the quantity $H_{AB}e^Ae^B$ has a geometric nature and should be expressible in terms of ${}^3R$, υ_{AB}, K_{AB}. After a lengthy calculation, we find a remarkable relation connecting this quantity with the evolution of the comoving mass, namely

$$2H_{AB}e^Ae^B\ell^4H=-6\left(\ell^4\mathcal{A}^2\right)^{\cdot}. \qquad (77)$$

It follows that

$$\begin{aligned}\left(\frac{\dot{\ell}}{\ell}\right)^{\cdot}+\left(\frac{\dot{\ell}}{\ell}\right)^2 &= -\frac{\ell'}{\ell}\dot{e}_A u^A-\mathcal{A}-\frac{1}{6M^3}p_{\parallel}-\\ &-\frac{\Lambda}{6M^3}-\alpha H^{-1}\left(\ell^4\mathcal{A}^2\right)^{\cdot}. \qquad (78)\end{aligned}$$

IV. THE EFFECT OF THE BULK ON THE BRANE EVOLUTION

In order to derive the brane cosmological evolution one must take into account the Israel-Darmois [34] junction conditions for the extrinsic curvature of the brane. As we have already mentioned, we impose a Z_2-symmetry around the location of the brane. We assume that the brane Universe is filled with a perfect fluid, so that the brane energy-momentum tensor takes the form

$$T^{\text{BRANE}}_{\alpha\beta}=\tilde{\rho}\tilde{u}_\alpha\tilde{u}_\beta+\tilde{p}\tilde{h}_{\alpha\beta} \qquad (79)$$

where $\tilde{h}_{\alpha\beta}=g_{\alpha\beta}+\tilde{u}_\alpha\tilde{u}_\beta$ is the projection tensor normally to the brane velocity $\tilde{u}_\alpha$. We note that *on* the brane the metric of the 3D screen space $\mathcal{D}$ coincides with $\tilde{h}_{\alpha\beta}$ so that $\Pi_{\alpha\beta}=\tilde{h}_{\alpha\beta}$ and $H=\dot{\ell}/\ell$ represents the Hubble parameter.

Essentially, equation (73) corresponds to the generalized Friedmann on the brane. From this equation, it can be seen that the discontinuity enters as the first derivative of the average length scale along e^A and is represented covariantly by the expansion of the spacelike congruence $\vartheta=\Pi^{\alpha\beta}e_{\alpha;\beta}=3\ell'/\ell$. Therefore, one should consider the junction conditions that involve only the spatial expansion i.e. the fully $\Pi-$projected of the first derivatives of e^A.

The covariant form of the junction conditions for braneworld models with a GB term in the bulk has been derived in [22]:

$$K_{\alpha\beta}+\frac{2\alpha}{3}[9J_{\alpha\beta}-2Jg_{\alpha\beta}-2\left(3P_{\alpha\gamma\beta\delta}+g_{\alpha\beta}G_{\gamma\delta}\right)K^{\gamma\delta}]=-\frac{1}{4M^3}\left(\tau_{\alpha\beta}-\frac{1}{3}\tau g_{\alpha\beta}\right), \qquad (80)$$

where

$$\begin{aligned}J_{\alpha\beta} &= \frac{1}{3}(2KK_{\alpha\gamma}K^{\gamma}{}_{\beta}+K_{\gamma\delta}K^{\gamma\delta}K_{\alpha\beta}-\\ &-2K_{\alpha\gamma}K^{\gamma\delta}K_{\delta\beta}-K^2K_{\alpha\beta}) \qquad (81)\end{aligned}$$

$$P_{\alpha\beta\gamma\delta}= {}^4R_{\alpha\beta\gamma\delta}+2g_{\alpha[\delta}{}^4R_{\gamma]\beta}+2g_{\beta[\gamma}{}^4R_{\delta]\alpha}+{}^4Rg_{\alpha[\gamma}g_{\delta]\beta}. \qquad (82)$$

Bearing in mind that $G_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta=3\left(H^2+k/\ell^2\right)$ [16], contracting (80) with $\Pi_{\alpha\beta}$, and using equations (6), (51) and (73), we get

$$\begin{aligned}&\frac{\ell'}{\ell}\left\{1+4\alpha\left[H^2+\frac{k}{\ell^2}-\frac{1}{3}\left(\frac{\ell'}{\ell}\right)^2\right]\right\}\\ &= -\frac{1}{12M^3}\left[V+\tilde{\rho}-6r_cM^3\left(H^2+\frac{k}{\ell^2}\right)\right], \qquad (83)\end{aligned}$$

or, equivalently,

$$\epsilon_2 \left[1 + \frac{8}{3}\alpha \left(H^2 + \frac{k}{\ell^2} + \frac{\mathcal{A}_0}{2}\right)\right] \left(H^2 + \frac{k}{\ell^2} - \mathcal{A}_0\right)^{1/2} = \\ = \frac{1}{12M^3}\left[(\tilde{\rho} + V) - 6r_c M^3 \left(H^2 + \frac{k}{\ell^2}\right)\right] \quad (84)$$

where $\epsilon_2 = \pm 1$ depends on whether the spacelike expansion ϑ (equivalently ℓ'/ℓ) is negative or positive in the vicinity of the brane. This is the same parameter that appears in equation (23). The quantity $\mathcal{A}_0 = \mathcal{A}(\tau, 0)$ is given by (75) and incorporates the effects of the bulk fluid [9]. Equation (84) is the generalization of the Friedmann equation on the brane given in [23]. As expected, the black hole mass $\mathcal{M}_0$ has been replaced by the comoving mass $\mathcal{M}$ of the bulk fluid [9].

Although equation (84) is a third-order polynomial in $\mathcal{B} = H^2 + k/\ell^2$, only *one* root is free of instabilities and reduces to the standard Randall-Sundrum solution. This can be seen by considering the pure induced-gravity model for which $\alpha = 0$. It follows from (84) that [16]

$$\frac{r_c^2}{2}\left(H^2 + \frac{k}{\ell^2}\right) = 1 + \frac{r_c\left(V + \tilde{\rho}\right)}{12M^3} - \\ -\epsilon_2 \left[1 + \frac{r_c\left(V + \tilde{\rho}\right)}{6M^3} + \frac{r_c^2 \Lambda}{12M^3} - \frac{r_c^2 \mathcal{M}}{6\pi^2 M^3 \ell^4}\right]^{1/2}. \quad (85)$$

The branch with $\epsilon_2 = -1$ reduces to the self-accelerating branch of the DGP cosmology [10, 14, 15] for $V = 0$. This branch is known to suffer from instabilities under small perturbations [17]. The value $\epsilon_2 = 1$ reproduces the normal stable branch of brane cosmology. For $\alpha \neq 0$, there are two possible values of $\mathcal{A}_0$, given by equation (75). The solution with $\epsilon_1 = -1$ reproduces the self-accelerating brane cosmology in the presence of a GB term. However, the bulk configuration is unstable in this case [25].

For completeness, we also give the *Raychaudhuri equation* in the presence of induced-gravity and GB terms. It follows from equation (78) and reads

$$\tilde{q} = -\frac{\Lambda}{6M^3} - \frac{\ell'}{\ell}\dot{e}_A u^A - \mathcal{A}_0 - \frac{1}{6M^3}p_\parallel - \alpha H^{-1}\left(\ell^4 \mathcal{A}_0^2\right)^{\cdot}, \quad (86)$$

where

$$\tilde{q} = \ddot{\ell}/\ell = \left(\dot{\ell}/\ell\right)^{\cdot} + H^2 \quad (87)$$

denotes the *acceleration parameter*. We note that, for arbitrary bulk matter configurations, the comoving mass depends on both the average length scale and the time scale defined by the brane observers $\tilde{u}^A$. However, for *on* brane considerations, we have $\mathcal{A}_0(\ell)$ and the acceleration parameter is written

$$\tilde{q} = -\frac{\Lambda}{6M^3} - \frac{\ell'}{\ell}\dot{e}_A u^A - \mathcal{A}_0 - \frac{1}{6M^3}p_\parallel - \alpha\ell\frac{d\left(\ell^4 \mathcal{A}_0^2\right)}{d\ell}. \quad (88)$$

The junction condition for the discontinuous quantity $\dot{e}_A u^A$ follows from (80)

$$0 = \left[8\frac{\ell'}{\ell}\tilde{q} - 4\mathcal{B}\cdot\left(\dot{e}_A u^A\right) + 4\left(\frac{\ell'}{\ell}\right)^2\left(\dot{e}_A u^A\right)\right]\alpha + \\ +2\frac{\ell'}{\ell} - \left(\dot{e}_A u^A\right) - \frac{1}{4M^3}\left[\tilde{p} - V + 2r_c^2 M^3\left(2\tilde{q} + \mathcal{B}\right)\right]. \quad (89)$$

We set

$$\Gamma = \frac{\Lambda}{6M^3} + \mathcal{A}_0 + \frac{1}{6M^3}p_\parallel + \alpha\ell\frac{d\left(\ell^4 \mathcal{A}_0^2\right)}{d\ell}, \quad (90)$$

where $p_\parallel = T_{AB}^{\text{BULK}} e^A e^B$ is the pressure in the preferred spacelike direction. By using equations (73), (83) and (86) we find

$$\tilde{q} = \frac{Q_1}{Q_2}, \quad (91)$$

where

$$Q_1 = 128\mathcal{A}_0^2\alpha(2\Gamma\alpha - 1) + 32\mathcal{A}_0[8\Gamma\alpha\left(2\mathcal{B}\alpha + 1\right) - \\ -4\mathcal{B}\alpha - 3] + 16\Gamma\left(8\mathcal{B}\alpha + 3\right) + 4\mathcal{B}^2\left(64\alpha - 3r_c^2\right) + \\ +2\mathcal{B}\left(48 + r_c\frac{4V - 3\tilde{p} + \tilde{\rho}}{M^3}\right) + \\ +\frac{1}{M^3}(\tilde{\rho} + V)(\tilde{p} - V) \quad (92)$$

$$Q_2 = 4\Big[64\mathcal{A}_0^2\alpha^2 + 32\mathcal{A}_0\alpha - 256\mathcal{B}^2\alpha^2 + \\ +2\mathcal{B}\left(3r_c^2 - 64\alpha\right) - 12 - r_c\frac{\tilde{\rho} + V}{M^3}\Big]. \quad (93)$$

Starting from equation (91) we can check which values of the parameters of the theory can lead to accelerating expansion. For example, if we restrict our considerations to the normal branch with $\epsilon_1 = \epsilon_2 = 1$, both the induced gravity model ($\alpha = 0$, $r_c \neq 0$) and the GB model ($\alpha \neq 0$, $r_c = 0$) require the standard mechanisms of inducing acceleration: either an effective cosmological constant, or a negative integrated mass of the bulk fluid, or a fluid with sufficiently negative pressure [9]. The full analysis of the various cosmological flows for all possible values of α, r_c, ϵ_1, ϵ_2 is beyond the scope of this work. For an AdS-Schwarzschild bulk, various possibilities have been studied in the literature. (See for example [28] and references therein.)

One significant property of the solutions with a nonzero bulk integrated mass $\mathcal{M}$ is that the brane energy density has extrema. This can be seen by considering $\tilde{\rho}$ as a function of $\mathcal{B}$ and $\mathcal{A}_0$

$$\tilde{\rho} + V = 2M^3\left[3r_c\mathcal{B} + 2\epsilon_2\sqrt{\mathcal{B} - \mathcal{A}_0}(8\alpha\mathcal{B} + 4\alpha\mathcal{A}_0 + 3)\right]. \quad (94)$$

We emphasize that if $\mathcal{M}=0$ the value of $\mathcal{A}_0$ is fixed by Λ and α (see eq. (75)), so that it does not vary with the scale ℓ and the following analysis is not valid. There exist extremal values

$$(\tilde{\rho}+V)_{\rm extr} = -\frac{M^3 r_c\left(-2r_c^2\epsilon_2+48\alpha-3r_c^2\right)}{32\alpha^2}, \qquad (95)$$

obtained for

$$\mathcal{B}=\frac{r_c^2-16\alpha}{64\alpha^2}, \quad \mathcal{A}_0=-\frac{1}{4\alpha}. \qquad (96)$$

For $\epsilon_2=1$ we get a *global* minimal value for the brane density

$$(\tilde{\rho}+V)_{\rm min} = \frac{M^3 r_c\left(5r_c^2-48\alpha\right)}{32\alpha^2}. \qquad (97)$$

For $\epsilon_2=-1$ we get a *global* maximal value

$$(\tilde{\rho}+V)_{\rm max} = \frac{M^3 r_c\left(r_c^2-48\alpha\right)}{32\alpha^2}. \qquad (98)$$

If we require a positive brane tension, it is clear that we must impose $r_c^2-48\alpha>0$ for a physically relevant solution.

V. CONCLUSIONS

The three equations (75), (84), (91) are the main results of our work. They determine the brane evolution for a given, but otherwise general, distribution of bulk matter. At the practical level, they demonstrate how to construct cosmological brane models in non-trivial bulk backgrounds. One first has to find a solution of the EFE in the bulk. The embedding of the brane is then automatic and leads to a cosmological evolution described by (75), (84) and (91). It must be emphasized that, from the point of view of the brane observer, the procedure described in the present paper leads to a definite prediction for the energy exchange rate. Assigning a physical meaning to this rate may not be always straightforward. In realistic cases the bulk configuration must have a certain freedom of parameters in order to accomodate the exchange rate that is expected on physical grounds. A typical example is provided by a brane that radiates or absorbs massless particles (Kaluza-Klein gravitons or gauge fields). The bulk metric is of the generalized Vaidya type and the energy-momentum tensor contains a radiation field. This metric includes an arbitrary function that can be fixed by requiring a rate of energy exchange consistent with the physical processes assumed on the brane (e.g. energy collisions in a hot plasma) [6]. We postpone the discussion of these issues in specific models for a future publication.

The possibility of accelerated expansion of the brane Universe is related to the structure of the Raychaudhuri equation (91). Its complicated nature does not permit a straightforward analysis. However, one can still draw some intuitive conclusions regarding the initial state of the brane Universe. Using equation (94) the expression for the acceleration parameter takes the simple form

$$\begin{aligned}\tilde{q} \;=\; & \{4M^3\left[2\mathcal{A}_0\left(2\Gamma\alpha-1\right)+\Gamma+2\mathcal{B}\right]+ \\ & +\epsilon_2\sqrt{\mathcal{B}-\mathcal{A}_0}\left(2r_cM^3\mathcal{B}+\tilde{p}-V\right)\}\times \\ & \times\left[4M^3\left(4\mathcal{A}_0\alpha-8\mathcal{B}\alpha-1-r_c\epsilon_2\sqrt{\mathcal{B}-\mathcal{A}_0}\right)\right]^{-1}.\end{aligned} \qquad (99)$$

For the values of $\mathcal{B}$ and $\mathcal{A}_0$ that correspond to the extrema of the brane energy density (equations (96)) the acceleration parameter becomes

$$\tilde{q}=-\frac{32M^3r_c\alpha+\left[M^3r_c\left(r_c^2-16\alpha\right)-32\alpha^2\left(V-\tilde{p}\right)\right]\epsilon_2}{128M^3r_c\alpha^2(1+\epsilon_2)}. \qquad (100)$$

In the branch with $\epsilon_2=-1$ the brane energy density has a maximal value given by equation (98). The standard picture of an initial singularity, accompanied by infinite energy density and deceleration, is replaced by a state of maximal energy density. However, from equation (100) we deduce that the acceleration parameter is infinite for *any* brane equation of state apart from $\tilde{p}=-\tilde{\rho}$. This signals the presence of a curvature singularity (see e.g. [28] for the simple case of a Minkowski bulk). On the other hand, if *initially* $\tilde{p}=-\tilde{\rho}$, using (95) we get

$$\tilde{q}=\frac{r_c^2-16\alpha}{64\alpha^2}. \qquad (101)$$

For $\alpha>0$ and under the constraint $r_c^2-48\alpha>0$ imposed at the end of the previous section, the brane evolution begins with finite energy density and pressure, as well as a *positive* and *finite* acceleration parameter. We point out that the elimination of the initial curvature singularity in the $\epsilon_2=-1$ branch depends crucially on the assumption $\mathcal{A}_0\neq 0$. This is only an example of the plethora of new phenomena that emerge from the combination of induced gravity on the brane, and a GB term and matter in the bulk.

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